

BOUNDEDNESS OF ANTI-CANONICAL VOLUMES OF SINGULAR LOG FANO THREEFOLDS

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ABSTRACT. We prove Weak Borisov–Alexeev–Borisov Conjecture in dimension three which states that the anti-canonical volume of an ϵ -klt log Fano pair of dimension three is bounded from above.

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1. INTRODUCTION

Throughout we work over the field of complex numbers \mathbb{C} . We adopt the standard notations and definitions in [17] and [23], and will freely use them.

Definition 1.1. A *pair* (X, Δ) consists of a normal projective variety X and an effective \mathbb{Q} -divisor Δ on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. (X, Δ) is called a *log Fano pair* (resp. *weak log Fano pair*) if $-(K_X + \Delta)$ is ample (resp. nef and big). If $\dim X = 2$, we will use *del Pezzo* instead of Fano.

Definition 1.2. Let (X, Δ) be a pair. Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) , write

$$K_Y = f^*(K_X + \Delta) + \sum a_i F_i,$$

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where F_i is a prime divisor. The coefficient a_i is called the *discrepancy* of F_i with respect to (X, Δ) , and denoted by $a_{F_i}(X, \Delta)$. For some $\epsilon \in [0, 1]$, the pair (X, Δ) is called

- (a) ϵ -kawamata log terminal (ϵ -klt, for short) if $a_i > -1 + \epsilon$ for all i ;
- (b) ϵ -log canonical (ϵ -lc, for short) if $a_i \geq -1 + \epsilon$ for all i ;
- (c) *terminal* if $a_i > 0$ for all f -exceptional divisors F_i .

Usually we write X instead of $(X, 0)$ in the case $\Delta = 0$.

Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense.

Definition 1.3. A variety X is of ϵ -Fano type if there exists an effective \mathbb{Q} -divisor Δ such that (X, Δ) is an ϵ -klt log Fano pair.

We are mainly interested in the boundedness of varieties of ϵ -Fano type.

Definition 1.4. A collection of varieties $\{X_\lambda\}_{\lambda \in \Lambda}$ is said to be *bounded* if there exists $h : \mathcal{X} \rightarrow S$ a projective morphism between schemes of finite type such that for each X_λ , $X_\lambda \simeq \mathcal{X}_s$ for some $s \in S$.

Our motivation is the following BAB Conjecture due to A. Borisov, L. Borisov, and V. Alexeev.

Conjecture 1.5 (BAB Conjecture). *Fix $0 < \epsilon < 1$, an integer $n > 0$.*

Then the set of all n -dimensional varieties of ϵ -Fano type is bounded.

BAB Conjecture is one of the most important conjecture in birational geometry and it is related to the termination of flips. As the approach to this conjecture, we are interested in the following much weak conjecture for anti-canonical volumes which is a consequence of BAB Conjecture.

Conjecture 1.6 (Weak BAB Conjecture). *Fix $0 < \epsilon < 1$, an integer $n > 0$.*

Then there exists a number $M(n, \epsilon)$ depending only on n and ϵ with the following property:

If X is an n -dimensional variety of ϵ -Fano type, then

$$\text{Vol}(-K_X) \leq M(n, \epsilon).$$

BAB Conjecture was proved in dimension two by Alexeev [1] with a simplified argument by Alexeev–Mori [2]. In dimension three or higher, BAB Conjecture is still open. There are only some partial boundedness results. For example, we have boundedness of smooth Fano manifolds by Kollár–Miyaoka–Mori [21], that of terminal \mathbb{Q} -Fano \mathbb{Q} -factorial threefolds of Picard number one by Kawamata [16], that of canonical \mathbb{Q} -Fano threefolds by Kollár–Miyaoka–Mori–Takagi [22], and that of toric varieties by Borisov–Borisov [7].

Weak BAB Conjecture in dimension two was treated by Alexeev [1], Alexeev–Mori [2], and Lai [24]. Recently, the author [15] gave an optimal value for the number $M(2, \epsilon)$ (see also Corollary 4.3). For Weak BAB Conjecture in dimension three assuming that Picard number of X is one, an effective value of $M(3, \epsilon)$ was announced by Lai [24]. For general case in dimension three and higher, Weak BAB Conjecture is still open.

As the main theorem of this paper, we prove Weak BAB Conjecture in dimension three.

Theorem 1.7. *Weak BAB Conjecture holds for $n = 3$.*

As a consequence, we get a different proof of a result on the boundedness of log Fano varieties of fixed index in dimension three which was conjectured by Batyrev, and proved by A. Borisov [6] in dimension three and Hacon–McKernan–Xu [12, Corollary 1.8] in arbitrary dimension.

Corollary 1.8. *Fix a positive integer r .*

Let \mathcal{D} be the set of all normal projective varieties X , where $\dim X = 3$, K_X is \mathbb{Q} -Cartier, and there exists an effective \mathbb{Q} -divisor Δ such that (X, Δ) is klt and $-r(K_X + \Delta)$ is Cartier and ample.

Then \mathcal{D} forms a bounded family.

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2. DESCRIPTION OF THE PROOF

Firstly, we give an approach to Weak BAB Conjecture via Mori fiber spaces.

Definition 2.1. A projective morphism $X \rightarrow T$ between normal varieties is called a *Mori fiber space* if the following conditions hold:

- (i) X is \mathbb{Q} -factorial with terminal singularities;
- (ii) f is a *contraction*, i.e., $f_*\mathcal{O}_X = \mathcal{O}_T$;
- (iii) $-K_X$ is ample over T ;
- (iv) $\rho(X/T) = 1$;
- (v) $\dim X > \dim T$.

At this time, we say that X is with a *Mori fiber structure*.

We raise the following conjecture for Mori fiber spaces.

Conjecture 2.2 (Weak BAB Conjecture for Mori fiber spaces). *Fix $0 < \epsilon < 1$, an integer $n > 0$.*

Then there exists a number $M(n, \epsilon)$ depending only on n and ϵ with the following property:

If X is an n -dimensional variety of ϵ -Fano type with a Mori fiber structure, then

$$\mathrm{Vol}(-K_X) \leq M(n, \epsilon).$$

We prove the following theorem by using Minimal Model Program.

Theorem 2.3. *Weak BAB Conjecture holds for fixed ϵ and n if and only if Weak BAB Conjecture for Mori fiber spaces holds for fixed ϵ and n .*

By Theorem 2.3, to consider the boundedness of anti-canonical volumes of log Fano pairs, we only need to consider the ones with better singularities (\mathbb{Q} -factorial terminal singularities) and with an additional structures (Mori fiber structures). This is the advantage of this theorem. In dimension two,

this theorem appears as a crucial step to get the optimal value of $M(2, \epsilon)$ (c.f. [15]).

Restricting our interest to dimension three, we prove the following theorem.

Theorem 2.4. *Weak BAB Conjecture for Mori fiber spaces holds for $n = 3$.*

Theorem 1.7 follows from Theorems 2.3 and 2.4 directly.

To prove Theorem 2.4, we need to consider 3-fold X of ϵ -Fano type with a Mori fiber structure $X \rightarrow T$. There are 3 cases:

- (1) $\dim T = 0$, X is a \mathbb{Q} -factorial terminal \mathbb{Q} -Fano 3-folds with $\rho = 1$;
- (2) $\dim T = 1$, $X \rightarrow T \simeq \mathbb{P}^1$ is a *del Pezzo fibration*, i.e. a general fiber is a smooth del Pezzo surface;
- (3) $\dim T = 2$, $X \rightarrow T$ is a *conic bundle*, i.e. a general fiber is a smooth rational curve.

The second statement is implied by the following fact: if (X, Δ) is a klt log Fano pair, then X is rationally connected (see [32, Theorem 1]), in particular, for any surjective morphism $X \rightarrow T$ to a normal curve, $T \simeq \mathbb{P}^1$.

In Case (1), X is bounded by Kawamata [16], and the optimal bound of $\text{Vol}(-K_X) = (-K_X)^3$ is 64 due to the classification on smooth Fano 3-folds of Iskovskikh and Mori–Mukai and by Namikawa’s result [28] (Gorenstein case) and Prokhorov [29] (non-Gorenstein case).

We will mainly treat Cases (2) and (3).

One basic idea is to construct singular pairs which is not klt along fibers of $X \rightarrow T$. Then by Connectedness Lemma, we may find a non-klt center intersecting with the fibers. Finally by restricting on a general fiber, we get the bound after some arguments on lower dimensional varieties. But several difficulties arise here.

In Case (3), the difficulty arises in the construction of singular pair because we need to avoid components which are vertical over T . To do this, we need a good understanding of the singularities and boundedness of the surface T , which was done by several papers as [1], [27], and [4].

In Case (2), the difficulty arises in the last step. After restricting on a general fiber, we need to bound the (generalized) log canonical thresholds on surfaces. So we are done by proving the following (generalized) Ambro’s conjecture in dimension two.

Definition 2.5. Let (X, B) be an lc pair and $D \geq 0$ be a \mathbb{Q} -Cartier \mathbb{Q} -divisor. The *log canonical threshold* of D with respect to (X, B) is

$$\text{lct}(X, B; D) = \sup\{t \in \mathbb{Q} \mid (X, B + tD) \text{ is lc}\}.$$

For the use of this paper, we need to consider the case when D is not effective. Let G be a \mathbb{Q} -Cartier \mathbb{Q} -divisor satisfying $G + B \geq 0$, The *generalized log canonical threshold* of G with respect to (X, B) is

$$\text{glct}(X, B; G) = \sup\{t \in [0, 1] \cap \mathbb{Q} \mid (X, B + tG) \text{ is lc}\}.$$

Note that the assumption $t \in [0, 1]$ guarantees that $B + tG \geq 0$.

Conjecture 2.6 (Ambro’s conjecture). *Fix $0 < \epsilon < 1$ and integer $n > 0$.*

Then there exists a number $\mu(n, \epsilon) > 0$ depending only on n and ϵ with the following property:

If (Y, B) is an ϵ -klt log Fano pair of dimension n , then

$$\inf\{\text{lct}(Y, B; D) \mid D \sim_{\mathbb{Q}} -(K_Y + B), D \geq 0\} \geq \mu(n, \epsilon).$$

Note that we do not assume any special conditions on the coefficients of B . The left-hand side of the inequality is called α -invariant of (Y, B) which generalizes the concept of α -invariant of Tian for Fano manifolds in differential geometry (see [8, 9, 31]). Recently Ambro [3] announced a proof of this conjecture assuming that (Y, B) is a toric pair where an explicit sharp number $\mu(n, \epsilon)$ was given. For the use of this paper, we need a stronger version of this conjecture where D may not be effective.

Conjecture 2.7 (generalized Ambro's conjecture). *Fix $0 < \epsilon < 1$ and integer $n > 0$.*

Then there exists a number $\mu(n, \epsilon) > 0$ depending only on n and ϵ with the following property:

If (Y, B) is an ϵ -klt weak log Fano pair of dimension n and Y has at worst terminal singularities, then

$$\inf\{\text{glct}(Y, B; G) \mid G \sim_{\mathbb{Q}} -(K_Y + B), G + B \geq 0\} \geq \mu(n, \epsilon).$$

Note that Conjecture 2.6 follows from Conjecture 2.7 easily after taking terminalization of (Y, B) .

We prove the conjecture in dimension two by following some ideas in the proof of BAB Conjecture in dimension two ([1, 2]). But it seems that this conjecture does not follow from BAB Conjecture trivially.

Theorem 2.8. *Conjecture 2.7 holds for $n = 2$.*

For the proof of Corollary 1.8, we basically follow the idea in [6] to bound the Hilbert polynomials by [20].

This paper is organized as follows. In Section 4, we prove the reduction step to Mori fiber spaces (Theorem 2.3). In Section 5, we prove generalized Ambro's conjecture in dimension two (Theorem 2.8). In Section 6, we prove Weak BAB Conjecture for Mori fiber spaces in dimension three (Theorem 2.4). In Section 7, we prove the boundedness of log Fano threefolds of fixed index (Corollary 1.8).

3. PRELIMINARIES

3.1. Volumes.

Definition 3.1. Let X be an n -dimensional projective variety and D be a Cartier divisor on X . The *volume* of D is the real number

$$\text{Vol}(D) = \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

Note that the limsup is actually a limit. Moreover by the homogenous property of the volume, we can extend the definition to \mathbb{Q} -Cartier \mathbb{Q} -divisors. Note that if D is a nef \mathbb{Q} -divisor, then $\text{Vol}(D) = D^n$. If D is a non- \mathbb{Q} -Cartier \mathbb{Q} -divisors, we may take a \mathbb{Q} -factorialization of X , i.e., a birational morphism $\phi : Y \rightarrow X$ which is isomorphic in codimension one and Y is \mathbb{Q} -factorial, then $\text{Vol}(D) := \text{Vol}(\phi_*^{-1}D)$. Note that \mathbb{Q} -factorialization always exists for klt pairs (cf. [5, Theorem 1.4.3]).

For more background on volumes, see [25, 11.4.A].

3.2. Hirzebruch surfaces.

We recall some basic properties of the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$. Denote by h (resp. f) the class in $\text{Pic } \mathbb{F}_n$ of the tautological bundle $\mathcal{O}_{\mathbb{F}_n}(1)$ (resp. of a fiber). Then $\text{Pic } \mathbb{F}_n = \mathbb{Z}h \oplus \mathbb{Z}f$ with $f^2 = 0$, $f \cdot h = 1$, $h^2 = n$. If $n > 0$, there is a unique irreducible curve $\sigma_n \subset \mathbb{F}_n$ such that $\sigma_n \sim h - nf$, $\sigma_n^2 = -n$. For $n = 0$, we can also choose one curve whose class in $\text{Pic } \mathbb{F}_0$ is h and denote it by σ_0 . Note that

$$-K_{\mathbb{F}_n} \sim 2h - (n - 2)f \sim 2\sigma_n + (n + 2)f.$$

Lemma 3.2. *For an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} -K_{\mathbb{F}_n}$ and a fiber f , $\text{mult}_f D \leq n + 2$.*

Proof. Since $D - (\text{mult}_f D)f$ is effective, $(D - (\text{mult}_f D)f) \cdot h \geq 0$. On the other hand, $(D - (\text{mult}_f D)f) \cdot h = n + 2 - \text{mult}_f D$. \square

Lemma 3.3. *Let $T = \mathbb{P}^2$ or \mathbb{F}_n , then for an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} -K_T$ and a point Q , $\text{mult}_Q D \leq n + 4$ holds. Moreover, if we write $D = \sum_j b_j D_j$ by its components and assume that $b_j \leq 1$ for all j , then $\sum_j b_j \leq 4$.*

Proof. If $T = \mathbb{P}^2$, taking a general line L through Q , we have

$$3 = (D \cdot L) \geq \text{mult}_Q(D).$$

If $T = \mathbb{F}_n$, take f be the fiber passing through Q , by Lemma 3.2 and intersection theory, we have

$$2 = D \cdot f \geq \text{mult}_Q D - \text{mult}_f D \geq \text{mult}_Q D - n - 2.$$

For the latter statement, if $T = \mathbb{F}_n$, then the conclusion follows by [2, Lemma 1.4]. If $T = \mathbb{P}^2$, then $\sum b_j \leq 3$ by degree computation. \square

3.3. Non-klt centers and connectedness lemma.

Definition 3.4. Let X be a normal projective variety and Δ be a \mathbb{Q} -divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) , write

$$K_Y = f^*(K_X + \Delta) + \sum a_i F_i,$$

where F_i is a prime divisor. F_i is called a *non-klt place* of (X, Δ) if $a_i \leq -1$. A subvariety $V \subset X$ is called a *non-klt center* of (X, Δ) if it is the image of a non-klt place. The *non-klt locus* $\text{Nklt}(X, \Delta)$ is the union of all non-klt centers of (X, Δ) . A non-klt center is *maximal* if it is an irreducible component of $\text{Nklt}(X, \Delta)$.

The following lemma suggests a standard way to construct non-klt centers.

Lemma 3.5 (cf. [23, Lemma 2.29]). *Let (X, Δ) be a pair and $Z \subset X$ be a close subvariety of codimension k such that Z is not contained in the singular locus of X . If $\text{mult}_Z \Delta \geq k$, then Z is a non-klt center of (X, Δ) .*

Recall that the *multiplicity* $\text{mult}_Z F$ of a divisor F along a subvariety Z is defined by the multiplicity $\text{mult}_x F$ of F at a general point $x \in Z$.

Unfortunately, the converse of Lemma 3.5 is not true unless $k = 1$. Usually we do not have good estimations for the multiplicity along a non-klt center but the following lemma.

Lemma 3.6 (cf. [25, Theorem 9.5.13]). *Let (X, Δ) be a pair and $Z \subset X$ be a non-klt center of (X, Δ) such that Z is not contained in the singular locus of X . Then $\text{mult}_Z \Delta \geq 1$.*

If we assume some simple normal crossing condition on the boundary, we can get more information on the multiplicity along a non-klt center. For simplicity, we just consider surfaces.

Lemma 3.7 (cf. [26, 4.1 Lemma]). *Fix $0 < e < 1$. Let S be a smooth surface, B be an effective \mathbb{Q} -divisor, and D be a (not necessarily effective) simple normal crossing supported \mathbb{Q} -divisor. Assume that coefficients of D are at most e and $\text{mult}_P B \leq 1 - e$ for some point P , then for arbitrary divisor E centered on P over S , $a_E(S, B + D) \geq -e$. In particular, if Z is a non-klt center of $(S, B + D)$ and coefficients of D are at most e , then $\text{mult}_Z B > 1 - e$.*

Proof. By taking a sequence of point blow-ups, we can get the divisor E . Consider the blow-up at P , we have $f : S_1 \rightarrow S$ with $K_{S_1} + B_1 + D_1 + mE_1 = f^*(K_S + B + D)$ where B_1 and D_1 are the strict transforms of B and D respectively, and E_1 is the exceptional divisor with $m = \text{mult}_P(B + D) - 1 \leq 1 - e + 2e - 1 = e$. Now $D_1 + mE_1$ is again simple normal crossing supported and $\text{mult}_Q B_1 \leq \text{mult}_P B$ for $Q \in E_1$. Hence by induction on the number of blow-ups, we conclude that the coefficient of E is at most e and hence $a_E(S, B + D) \geq -e$. \square

We have the following connectedness lemma of Kollár and Shokurov for non-klt locus (cf. Shokurov [30], Kollár [19, 17.4]).

Theorem 3.8 (Connectedness Lemma). *Let $f : X \rightarrow Z$ be a proper morphism of normal varieties with connected fibers and D is a \mathbb{Q} -divisor such that $-(K_X + D)$ is \mathbb{Q} -Cartier, f -nef and f -big. Write $D = D^+ - D^-$ where D^+ and D^- are effective with no common components. If D^- is f -exceptional (i.e. all of its components have image of codimension at least 2), then $\text{Nklt}(X, D) \cap f^{-1}(z)$ is connected for any $z \in Z$.*

Remark 3.9. There are two main cases of interest of Connectedness Lemma:

- (i) Z is a point and (X, D) is a weak log Fano pair. Then $\text{Nklt}(X, D)$ is connected.
- (ii) $f : X \rightarrow Z$ is birational, (Z, B) is a log pair and $K_X + D = f^*(K_Z + B)$.

4. REDUCTION TO MORI FIBER SPACES

In this section, we prove the reduction step to Mori fiber spaces (Theorem 2.3).

The “only if” direction is trivial, we only need to prove the “if” direction.

Fix $0 < \epsilon < 1$, an integer $n > 0$. Let (X, Δ) be an ϵ -klt log Fano pair of dimension n . By [5, Corollary 1.4.3], taking \mathbb{Q} -factorialization of (X, Δ) , we have $\phi : X_0 \rightarrow X$ where $K_{X_0} + \phi_*^{-1}\Delta = \phi^*(K_X + \Delta)$, X_0 is \mathbb{Q} -factorial, and ϕ is isomorphic in codimension one. Note that $\text{Vol}(-K_{X_0}) = \text{Vol}(-K_X)$.

Again by [5, Corollary 1.4.3], taking terminalization of X_1 , we have $\pi : X_1 \rightarrow X_0$ where $K_{X_1} + \Delta_{X_1} = \pi^*(K_{X_0} + \phi_*^{-1}\Delta)$, Δ_{X_1} is an effective \mathbb{Q} -divisor,

X_1 is \mathbb{Q} -factorial terminal, and (X_1, Δ_{X_1}) is ϵ -klt. Here $-(K_{X_1} + \Delta_{X_1})$ is nef and big.

By Kodaira's lemma (cf. [23, Proposition 2.61]) there exist a \mathbb{Q} -divisor Δ_1 such that $\Delta_1 \geq \Delta_{X_1}$, $-(K_{X_1} + \Delta_1)$ is ample, and (X, Δ_1) is ϵ -klt. In particular, X_1 is \mathbb{Q} -factorial terminal and of ϵ -Fano type.

Running K -MMP on X_1 , we get a sequence of normal projective varieties:

$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots \dashrightarrow X_r \rightarrow T.$$

Since $-K_{X_1}$ is big, this sequence ends up with a Mori fiber space $X_r \rightarrow T$ (cf. [5, Corollary 1.3.3]). In particular, X_r is \mathbb{Q} -factorial terminal.

Being of ϵ -Fano type is preserved by MMP according to the following lemma.

Lemma 4.1 (cf. [11, Lemma 3.1]). *Let Y be a projective normal variety and $f : Y \rightarrow Z$ be a projective birational contraction.*

- (1) *If Y is of ϵ -Fano type, so is Z ;*
- (2) *Assume that f is small, then Y is of ϵ -Fano type if and only if so is Z .*

In particular, minimal model program preserves ϵ -Fano type.

Proof. The proof is almost the same as [11, Lemma 3.1] where 0-Fano type is considered. First we assume that Y is of ϵ -Fano type, that is, there exists an effective \mathbb{Q} -divisor Δ on Y such that (Y, Δ) is ϵ -klt log Fano pair. Let H be a general effective ample divisor on Z and take a sufficiently small rational number $\delta > 0$ such that $-(K_Y + \Delta + \delta f^*H)$ is ample and $(Y, \Delta + \delta f^*H)$ is ϵ -klt. Then take a general effective ample \mathbb{Q} -divisor A on Y such that $(Y, \Delta + \delta f^*H + A)$ is ϵ -klt and

$$K_Y + \Delta + \delta f^*H + A \sim_{\mathbb{Q}} 0.$$

Then

$$K_Z + f_*\Delta + \delta H + f_*A = f_*(K_Y + \Delta + \delta f^*H + A) \sim_{\mathbb{Q}} 0,$$

and

$$f^*(K_Z + f_*\Delta + \delta H + f_*A) = K_Y + \Delta + \delta f^*H + A.$$

Therefore, $(Z, f_*\Delta + \delta H + f_*A)$ is ϵ -klt. Hence $(Z, f_*\Delta + f_*A)$ is ϵ -klt and $-(K_Z + f_*\Delta + f_*A) \sim_{\mathbb{Q}} \delta H$ is ample, that is, Z is of ϵ -Fano type.

Next we assume that f is small and Z is of ϵ -Fano type. Let Γ be an effective \mathbb{Q} -divisor on Z such that (Z, Γ) is ϵ -klt log Fano pair. Let Γ_Y be the strict transform of Γ on Y . Since f is small,

$$K_Y + \Gamma_Y = f^*(K_Z + \Gamma).$$

Hence (Y, Γ_Y) is ϵ -klt and $-(K_Y + \Gamma_Y)$ is nef and big. By Kodaira's lemma, there exist a \mathbb{Q} -divisor Γ' such that $\Gamma' \geq \Gamma_Y$, $-(K_Y + \Gamma')$ is ample and (Y, Γ') is ϵ -klt, that is, Y is of ϵ -Fano type.

We proved the lemma. \square

By Lemma 4.1, for all i , X_i is of ϵ -Fano type. To compare the volumes between these varieties, we have the following lemma.

Lemma 4.2. *Let $X_i \dashrightarrow X_{i+1}$ be one step of K -MMP. Then*

$$\text{Vol}(-K_{X_i}) \leq \text{Vol}(-K_{X_{i+1}}).$$

Proof. Take a common resolution $p : W \rightarrow X_i$, $q : W \rightarrow X_{i+1}$. Then

$$p^*(K_{X_i}) = q^*(K_{X_{i+1}}) + E,$$

where E is an effective q -exceptional \mathbb{Q} -divisor. Hence

$$\begin{aligned} \text{Vol}(-K_{X_i}) &= \text{Vol}(-p^*(K_{X_i})) \\ &= \text{Vol}(-q^*(K_{X_{i+1}}) - E) \\ &\leq \text{Vol}(-q^*(K_{X_{i+1}})) \\ &= \text{Vol}(-K_{X_{i+1}}). \end{aligned}$$

We proved the lemma. \square

Therefore we can compare the volumes on X and X_r . Recall that we take X_1 as the terminalization of X_0 , we have $K_{X_1} + F = \pi^*K_{X_0}$ with F an effective \mathbb{Q} -divisor. Hence

$$\text{Vol}(-K_{X_0}) \leq \text{Vol}(-K_{X_1}).$$

By Lemma 4.2,

$$\begin{aligned} \text{Vol}(-K_X) &= \text{Vol}(-K_{X_0}) \\ &\leq \text{Vol}(-K_{X_1}) \\ &\leq \text{Vol}(-K_{X_r}). \end{aligned}$$

Now X_r is an n -dimensional variety of ϵ -Fano type with a Mori fiber structure by construction. Assuming Weak BAB Conjecture for Mori fiber spaces, there exists $M(n, \epsilon)$ such that

$$\text{Vol}(-K_{X_r}) \leq M(n, \epsilon).$$

Hence

$$\text{Vol}(-K_X) \leq M(n, \epsilon).$$

We complete the proof of Theorem 2.3.

As a direct corollary, we recover the main result in [15] on Weak BAB Conjecture in dimension two.

Corollary 4.3. *Fix $0 < \epsilon < 1$.*

Then there exists a number

$$M(2, \epsilon) := \max \left\{ 9, \lfloor 2/\epsilon \rfloor + 4 + \frac{4}{\lfloor 2/\epsilon \rfloor} \right\}$$

with the following property:

If X is a surface of ϵ -klt del Pezzo type, then

$$\text{Vol}(-K_X) \leq M(2, \epsilon).$$

Proof. By Theorem 2.3, we only need to consider the cases when $X = \mathbb{P}^2$ or \mathbb{F}_n with $n \leq 2/\epsilon$ (see [2, Lemma 1.4] or [15, Lemma 3.1]). And the result follows by volume computation directly. \square

5. GENERALIZED AMBRO'S CONJECTURE IN DIMENSION TWO

In this section, we prove generalized Ambro's conjecture in dimension two (Theorem 2.8).

Fix an ϵ -klt weak log del Pezzo pair (S, B) with S smooth and a \mathbb{Q} -divisor $G \sim_{\mathbb{Q}} -(K_S + B)$ such that $G + B \geq 0$. Set $a := \text{glct}(S, B; G)$. Since we work on \mathbb{Q} -divisors, a is a positive rational number. The problem is to bound a from below. We may assume that $a < 1$. Set $D = G + B \geq 0$. Then $(S, B + aG) = (S, (1 - a)B + aD)$ is not klt. Note that $D \sim_{\mathbb{Q}} -K_S$.

By Base Point Free Theorem (cf. [23, Theorem 3.3]), $-(K_S + B)$ is semi-ample. Hence there exists an effective \mathbb{Q} -divisor M such that $K_S + B + M \sim_{\mathbb{Q}} 0$ and $(S, B + M)$ is ϵ -klt. For any birational morphism $f : S \rightarrow T$ between smooth surfaces, we have

$$K_S + B + M = f^*(K_T + f_*B + f_*M),$$

$$K_S + (1 - a)(B + M) + aD = f^*(K_T + (1 - a)(f_*B + f_*M) + af_*D).$$

Hence $(T, f_*B + f_*M)$ is ϵ -klt and $(T, (1 - a)(f_*B + f_*M) + af_*D)$ is not klt with

$$K_T + f_*B + f_*M \sim_{\mathbb{Q}} K_T + (1 - a)(f_*B + f_*M) + af_*D \sim_{\mathbb{Q}} 0.$$

Recall that either $S \simeq \mathbb{P}^2$ or there exists a birational morphism $g : S \rightarrow \mathbb{F}_n$ with $n \leq 2/\epsilon$ by [2, Lemma 1.4] or [15, Lemma 3.1].

Hence by replacing S by $T = \mathbb{P}^2$ or \mathbb{F}_n , we may assume that there exists a triple (T, B_T, D_T) satisfying the following conditions:

- (i) $T = \mathbb{P}^2$ or \mathbb{F}_n with $n \leq 2/\epsilon$;
- (ii) B_T, D_T are effective \mathbb{Q} -divisors on T ;
- (iii) (T, B_T) is ϵ -klt and $(T, (1 - a)B_T + aD_T)$ is not klt;
- (iv) $K_T + B_T \sim_{\mathbb{Q}} K_T + (1 - a)B_T + aD_T \sim_{\mathbb{Q}} 0$, equivalently, $B_T \sim_{\mathbb{Q}} D_T \sim_{\mathbb{Q}} -K_T$.

Since $(T, (1 - a)B_T + aD_T)$ is not klt, we may take a sequence of point blow-ups

$$T_{r+1} \rightarrow T_r \rightarrow \cdots \rightarrow T_2 \rightarrow T_1 = T$$

where $T_{i+1} \rightarrow T_i$ is the blow-up at a non-klt center $P_i \in \text{Nklt}(T_i, (1 - a)B_i + aD_i + E_i)$ where B_i and D_i are the strict transforms of B_T and D_T respectively and

$$K_{T_i} + (1 - a)B_i + aD_i + E_i = \pi_i^*(K_T + (1 - a)B_T + aD_T),$$

where $\pi_i : T_i \rightarrow T$ is the composition map and E_i is a π_i -exceptional \mathbb{Q} -divisor. We stop this process at T_{r+1} if

$$\dim \text{Nklt}(T_{r+1}, (1 - a)B_{r+1} + aD_{r+1} + E_{r+1}) > 0.$$

Since P_i is a non-klt center of $(T_i, (1 - a)B_i + aD_i + E_i)$, $\text{mult}_{P_i}((1 - a)B_i + aD_i + E_i) \geq 1$. Note that the coefficients of E_i are $(\text{mult}_{P_j}((1 - a)B_j + aD_j + E_j) - 1)$ for $j < i$, hence E_i is effective for all i . Furthermore, we may assume that $\text{mult}_{P_i}B_i$ is non-increasing. Take the integer $k \leq r$ such that $\text{mult}_{P_i}B_i \geq \epsilon/2$ for $i \leq k$ and $\text{mult}_{P_i}B_i < \epsilon/2$ for $i > k$. Write $B_T = \sum_j b_j B^j$ and $B_i = \sum_j b_j B_i^j$ by components. We have $b_j < 1 - \epsilon$ since (T, B_T) is ϵ -klt. Recall that $\sum_j b_j \leq 4$ by Lemma 3.3.

Claim 1. *If $\text{mult}_{B^j}(aD_T) > \epsilon/2$ for some j , then $a \geq \epsilon^2/(4 + 4\epsilon)$.*

Proof. Recall that $T = \mathbb{P}^2$ or \mathbb{F}_n with $n \leq 2/\epsilon$.

If $T = \mathbb{P}^2$, then $\text{mult}_{B^j} D_T \leq 3$ by degree counting. If $T = \mathbb{F}_n$ and B^j is a fiber, then $\text{mult}_{B^j} D_T \leq n + 2 \leq 2/\epsilon + 2$ by Lemma 3.2. If $T = \mathbb{F}_n$ and B^j is not a fiber, then $\text{mult}_{B^j} D_T \leq D_T \cdot f = 2$ where f is a fiber. Hence

$$a \geq \frac{\epsilon}{2\text{mult}_{B^j} D_T} \geq \frac{\epsilon^2}{4 + 4\epsilon}.$$

We proved the claim. \square

Since we need a lower bound of a , from now on, we may assume that $\text{mult}_{B^j}(aD_T) \leq \epsilon/2$ for all j . In particular, $\text{mult}_{B^i}(aD_i) \leq \epsilon/2$ and

$$\text{mult}_{B^i}((1-a)B_i + aD_i) < 1 - \epsilon/2$$

for all i and j .

Claim 2. *$(B_{k+1}^j)^2 \geq -4/\epsilon$ for all j .*

Proof. If $(B_{k+1}^j)^2 < 0$, then

$$\begin{aligned} -2 &\leq 2p_a(B_{k+1}^j) - 2 = (K_{T_{k+1}} + B_{k+1}^j) \cdot B_{k+1}^j \\ &= \frac{\epsilon}{2}(B_{k+1}^j)^2 + (K_{T_{k+1}} + (1 - \frac{\epsilon}{2})B_{k+1}^j) \cdot B_{k+1}^j \\ &\leq \frac{\epsilon}{2}(B_{k+1}^j)^2 + (K_{T_{k+1}} + (1-a)B_{k+1} + aD_{k+1} + E_{k+1}) \cdot B_{k+1}^j \\ &= \frac{\epsilon}{2}(B_{k+1}^j)^2 < 0, \end{aligned}$$

Hence we proved the claim. \square

Now we can bound the number k . On T_{k+1} , we have

$$\begin{aligned} (B_{k+1})^2 &= \left(\sum_j b_j B_{k+1}^j\right)^2 \geq \sum_j b_j^2 (B_{k+1}^j)^2 \geq \left(\sum_j b_j^2\right)(-4/\epsilon) \\ &\geq \left(\sum_j b_j\right)(1-\epsilon)(-4/\epsilon) \geq 16 - \frac{16}{\epsilon} \end{aligned}$$

and $(B_1)^2 = (K_T)^2 \leq 9$. On the other hand, at each blow-up, $(B_i)^2$ decreases by at least $\epsilon^2/4$ by the assumption $\text{mult}_{P_i} B_i \geq \epsilon/2$ for $i \leq k$. Hence

$$k \leq \frac{9 - (16 - 16/\epsilon)}{\epsilon^2/4} \leq \frac{64}{\epsilon^3}.$$

Now we consider $\pi_{k+1}^*(aD_T)$ on T_{k+1} .

Claim 3. *There exists a point Q on T_{k+1} such that $\text{mult}_Q \pi_{k+1}^*(aD_T) \geq \epsilon/4$.*

Proof. Consider the pair $(T_{k+1}, (1-a)B_{k+1} + aD_{k+1} + E_{k+1})$. Note that E_{k+1} is simple normal crossing supported.

Assume that there exists a curve E with coefficient at least $1 - 3\epsilon/4$ in E_{k+1} , that is,

$$\text{mult}_E(K_{T_{k+1}} - \pi_{k+1}^*(K_T + (1-a)B_T + aD_T)) \leq -1 + 3\epsilon/4.$$

On the other hand, since $(T, (1-a)B_T)$ is ϵ -klt,

$$\text{mult}_E(K_{T_{k+1}} - \pi_{k+1}^*(K_T + (1-a)B_T)) > -1 + \epsilon.$$

Hence $\text{mult}_E \pi_{k+1}^*(aD_T) \geq \epsilon/4$.

If all coefficients of E_{k+1} are smaller than $1 - 3\epsilon/4$, then $k < r$ and P_{k+1} is a non-klt center of $(T_{k+1}, (1-a)B_{k+1} + aD_{k+1} + E_{k+1})$. By Lemma 3.7, $\text{mult}_{P_{k+1}}((1-a)B_{k+1} + aD_{k+1}) \geq 3\epsilon/4$. Then $\text{mult}_{P_{k+1}}(aD_{k+1}) \geq \epsilon/4$ since $\text{mult}_{P_{k+1}}B_{k+1} < \epsilon/2$ by assumption. In particular, $\text{mult}_{P_{k+1}} \pi_{k+1}^*(aD_T) \geq \text{mult}_{P_{k+1}}(aD_{k+1}) \geq \epsilon/4$.

We proved the claim. \square

Now we will estimate $\text{mult}_{Q_1}(aD_T)$ where Q_1 is the image of Q on T . By removing unnecessary blow-ups, we may assume that we have a sequence of blow-ups

$$T_{k+1} \rightarrow T_k \rightarrow \cdots \rightarrow T_2 \rightarrow T_1 = T$$

where $f_{i+1} : T_{i+1} \rightarrow T_i$ is the blow-up at Q_i which is the image of Q on T_i with $k \leq 64/\epsilon^3$. Recall that $\pi_i : T_i \rightarrow T$ is the composition map and D_i is the strict transform of D_T on T_i . Denote C_{i+1} to be the exceptional divisor of f_{i+1} and C_{i+1}^j be its strict transform on T_j for $j \geq i+1$. We can write

$$\pi_j^*(aD_T) = aD_j + \sum_{2 \leq i \leq j} c_i C_i^j,$$

with $c_i = \text{mult}_{Q_{i-1}} \pi_{i-1}^*(aD_T)$.

Claim 4. *If $\text{mult}_{Q_1}(aD_T) \leq \alpha$, then $\text{mult}_{Q_i} \pi_i^*(aD_T) \leq (F_{i+1} - 1)\alpha$ for $1 \leq i \leq k+1$. Here F_n is the Fibonacci number with relation $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ and $F_0 = F_1 = 1$.*

Proof. We run induction on i . The case $i = 1$ is trivial. Assume the conclusion holds for $i < j$, then noting that Q_j is contained in at most two exceptional curves, we have

$$\begin{aligned} \text{mult}_{Q_j} \pi_j^*(aD_T) &= \text{mult}_{Q_j}(aD_j + \sum_{2 \leq i \leq j} c_i C_i^j) \\ &\leq \text{mult}_{Q_j}(aD_j) + (F_j - 1)\alpha + (F_{j-1} - 1)\alpha \\ &\leq \text{mult}_{Q_1}(aD_1) + (F_j - 1)\alpha + (F_{j-1} - 1)\alpha \\ &\leq (F_{j+1} - 1)\alpha. \end{aligned}$$

We proved the claim. \square

By Claims 3 and 4, $\text{mult}_{Q_1}(aD_T) \geq \epsilon/(4F_{k+2} - 4)$. Recall that $D_T \sim_{\mathbb{Q}} -K_T$ and $T = \mathbb{P}^2$ or \mathbb{F}_n with $n \leq 2/\epsilon$. By Lemma 3.3, $\text{mult}_{Q_1}(D_T) \leq n + 4$, combining with the inequality $k \leq 64/\epsilon^3$, we have

$$a \geq \frac{\epsilon^2}{(2 + 4\epsilon)(4F_{\lfloor 64/\epsilon^3 \rfloor + 2} - 4)},$$

and hence we may take this number to be $\mu(2, \epsilon)$.

We have proved Theorem 2.8.

6. WEAK BAB CONJECTURE FOR MORI FIBER SPACES IN DIMENSION THREE

In this section, we prove Weak BAB Conjecture for Mori fiber spaces in dimension 3 (Theorem 2.4). Recall that by a Mori fiber space we always mean a \mathbb{Q} -factorial terminal one.

Fix $0 < \epsilon < 1$ and consider an ϵ -klt log Fano pair (X, Δ) of dimension 3 with a Mori fiber structure. As explained, there are three cases:

- (1) X is a \mathbb{Q} -factorial terminal \mathbb{Q} -Fano 3-folds with $\rho = 1$;
- (2) $X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration;
- (3) $X \rightarrow S$ is a conic bundle.

As mentioned before, Case (1) is done by Kawamata [16]. We treat Cases (2) and (3) in the following two subsections.

6.1. Contractions to a curve.

In this subsection, we treat the case under a more general setting when there is a contraction $f : X \rightarrow \mathbb{P}^1$ (e.g. X has a del Pezzo fibration structure).

Theorem 6.1. *Let (Y, B) be an ϵ -klt log Fano pair of dimension n with a contraction $g : Y \rightarrow \mathbb{P}^1$ and Y having terminal singularities. Assume that Weak BAB Conjecture and generalized Ambro's conjecture hold in dimension $n-1$ with $M(n-1, \epsilon)$ and $\mu(n-1, \epsilon)$ the numbers defined in these conjectures. Then*

$$\text{Vol}(-K_Y) \leq \frac{2nM(n-1, \epsilon)}{\mu(n-1, \epsilon)}.$$

Proof. Note that Y is terminal by assumption. Hence a general fiber F of g is terminal and of ϵ -Fano type of dimension $n-1$ by adjunction formula. In particular, K_F is \mathbb{Q} -Cartier. It follows that $\text{Vol}(-K_F) \leq M(n-1, \epsilon)$ by Weak BAB Conjecture in dimension $n-1$.

Fix a rational number s satisfying

$$\frac{\text{Vol}(-K_Y)}{nM(n-1, \epsilon)} - \frac{1}{A} < s < \frac{\text{Vol}(-K_Y)}{nM(n-1, \epsilon)}$$

for sufficiently large number A . To bound $\text{Vol}(-K_Y)$ from above, it is sufficient to bound s from above. And we may assume that $s > 2$.

The following lemma allows us to construct non-klt centers.

Claim 5. *For a general fiber F of g , $-K_Y - sF$ is \mathbb{Q} -effective. In particular, there exists an effective \mathbb{Q} -divisor $B_F \sim_{\mathbb{Q}} -\frac{1}{s}K_Y$ such that F is a non-klt center of (Y, B_F) .*

Proof. For a positive integer p and a sufficiently divisible positive integer m , we have exact sequence

$$0 \rightarrow \mathcal{O}_Y(-mK_Y - pF) \rightarrow \mathcal{O}_Y(-mK_Y - (p-1)F) \rightarrow \mathcal{O}_F(-mK_Y - (p-1)F) \rightarrow 0.$$

Note that $\mathcal{O}_F(-mK_Y - (p-1)F) = \mathcal{O}_F(-mK_F)$. Hence

$$h^0(Y, \mathcal{O}_Y(-mK_Y - pF)) \geq h^0(Y, \mathcal{O}_Y(-mK_Y - (p-1)F)) - h^0(F, \mathcal{O}_F(-mK_F)).$$

Inductively, we have

$$h^0(Y, \mathcal{O}_Y(-mK_Y - pF)) \geq h^0(Y, \mathcal{O}_Y(-mK_Y)) - ph^0(F, \mathcal{O}_F(-mK_F)).$$

We may take $p = sm$ since m is sufficiently divisible. By the definition of volume, we have

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \frac{h^0(Y, \mathcal{O}_Y(-mK_Y)) - smh^0(F, \mathcal{O}_F(-mK_F))}{m^n} \\ &= \frac{1}{n!} \text{Vol}(-K_Y) - \frac{s}{(n-1)!} \text{Vol}(-K_F) > 0. \end{aligned}$$

Hence $h^0(Y, \mathcal{O}_Y(-mK_Y - smF)) > 0$ for m sufficiently divisible, that is, $-K_Y - sF$ is \mathbb{Q} -effective. In particular, there exists an effective \mathbb{Q} -divisor $B_F \sim_{\mathbb{Q}} -\frac{1}{s}K_Y$ such that $B_F - F \geq 0$, and hence F is a non-klt center of (Y, B_F) .

We proved the claim. \square

Now for two general fibers F_1 and F_2 , consider $B' = B_{F_1} + B_{F_2}$. By construction, $F_1 \cup F_2 \subset \text{Nklt}(Y, (1 - \frac{2}{s})B + B')$. Note that

$$-(K_Y + (1 - \frac{2}{s})B + B') \sim_{\mathbb{Q}} -(1 - \frac{2}{s})(K_Y + B)$$

is ample, by Connectedness Lemma, $\text{Nklt}(Y, (1 - \frac{2}{s})B + B')$ is connected. Hence there is a non-klt center $W \subset \text{Nklt}(Y, (1 - \frac{2}{s})B + B')$ connecting F_1 and F_2 . In particular, W dominates \mathbb{P}^1 . Restricting on a general fiber F , by adjunction formula, we have $(F, B|_F)$ is ϵ -klt log Fano with F terminal and $(F, (1 - \frac{2}{s})B|_F + B'|_F)$ is not klt (see [23, Lemma 5.17, Lemma 5.50]) with $B'|_F \sim_{\mathbb{Q}} -\frac{2}{s}K_F$. Hence

$$\frac{2}{s} \geq \text{glct}(F, B|_F; \frac{s}{2}B'|_F - B|_F).$$

To bound s from above, generalized Ambro's conjecture arises naturally. By generalized Ambro's conjecture in dimension $n - 1$,

$$s \leq \frac{2}{\mu(n-1, \epsilon)},$$

and hence

$$\text{Vol}(-K_Y) \leq \frac{2nM(n-1, \epsilon)}{\mu(n-1, \epsilon)}.$$

We completed the proof. \square

In particular, by Corollary 4.3 and Theorem 2.8, Weak BAB Conjecture and generalized Ambro's conjecture hold in dimension 2, and hence the following corollary holds.

Corollary 6.2. *Let (X, Δ) be an ϵ -klt log Fano pair of dimension 3 with a contraction $f : X \rightarrow \mathbb{P}^1$ and X having terminal singularities. Then*

$$\text{Vol}(-K_X) \leq \frac{6M(2, \epsilon)}{\mu(2, \epsilon)}.$$

6.2. Conic bundles.

In this subsection, we treat the case that X has a conic bundle structure $f : X \rightarrow S$. Firstly we collect some facts about singularities of the surface S .

Theorem 6.3. *Let (X, Δ) be an ϵ -klt log Fano pair of dimension 3 and $f : X \rightarrow S$ be a Mori fiber space to a surface S , then*

- (i) S has only Du Val singularities of type A ;
- (ii) There exists an effective \mathbb{Q} -divisor Δ_S on S , such that (S, Δ_S) is klt log del Pezzo;
- (iii) S is a Mori dream space;
- (iv) There exists an effective \mathbb{Q} -divisor Δ'_S on S , such that (S, Δ'_S) is $\delta(\epsilon)$ -klt and $K_S + \Delta'_S \sim_{\mathbb{Q}} 0$, where $\delta(\epsilon)$ is a positive real number depending only on ϵ ;
- (v) The family of such S is bounded, in particular, the Picard number of minimal resolution of S is bounded by $128/\delta(\epsilon)^5$.
- (vi) S is $N(\epsilon)$ -factorial, i.e. for a Weil divisor D on S , $N(\epsilon)D$ is Cartier, where $N(\epsilon)$ is a positive integer depending only on ϵ .

Proof. (i) is by [27, (1.2.7) Theorem]. (ii) is by [10, Corollary 3.3]. (iii) is by (ii) and [5, Corollary 1.3.2]. (iv) is by [4, Corollary 1.7] since we may find a boundary $\Delta' \geq \Delta$ such that (X, Δ') is ϵ -klt and $K_X + \Delta' \sim_{\mathbb{Q}} 0$. (v) is by (iv), [1, Theorem 6.8], and [2, Theorem 1.8]. (vi) is a direct consequence of (i) and (v). \square

For the definition and properties of *Mori dream spaces* we refer to the famous paper by Hu–Keel [14]. We will use the following property of Mori dream spaces: every nef divisor on S is semi-ample and there are finitely many irreducible curves with negative self intersection. In particular, a curve through a general point is nef. By a curve we always mean an irreducible reduced one.

If there is a curve C on S satisfying $(C)^2 = 0$, then C is semi-ample. In particular, a multiple of C induces a contraction $S \rightarrow \mathbb{P}^1$ and we are done by Subsection 6.1. Hence we may assume that there does not exist such curve C on S satisfying $(C)^2 = 0$.

Fix a positive rational number t satisfying

$$\frac{\text{Vol}(-K_X)}{24} - \frac{1}{A} < t^2 < \frac{\text{Vol}(-K_X)}{24}$$

for sufficiently large number A . To bound $\text{Vol}(-K_X)$ from above, it is sufficient to bound t from above. And we may assume that $t > 768N(\epsilon)/\epsilon$.

Lemma 6.4. *For a general fiber F of f ,*

$$h^0(X, \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^{2tm}) > 0$$

for m sufficiently divisible, where \mathcal{I}_F is the ideal sheaf of F . In particular, there exists an effective \mathbb{Q} -divisor $\Delta_F \sim_{\mathbb{Q}} -\frac{1}{t}K_X$ such that $\text{mult}_F \Delta_F \geq 2$.

Proof. For a positive integer p and a sufficiently divisible positive integer m , we have exact sequence

$$0 \rightarrow \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^p \rightarrow \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^{p-1} \rightarrow \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^{p-1} / \mathcal{I}_F^p \rightarrow 0.$$

Note that $\mathcal{I}_F^{p-1}/\mathcal{I}_F^p = S^{p-1}(\mathcal{I}_F/\mathcal{I}_F^2)$ (see [13, II. Theorem 8.24]) and $\mathcal{I}_F/\mathcal{I}_F^2 = \mathcal{O}_F^{\oplus 2}$. Hence

$$h^0(X, \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^p) \geq h^0(X, \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^{p-1}) - ph^0(F, \mathcal{O}_F(-mK_F)).$$

Inductively, we have

$$h^0(X, \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^p) \geq h^0(X, \mathcal{O}_X(-mK_X)) - \frac{p(p+1)}{2} h^0(F, \mathcal{O}_F(-mK_F)).$$

We may take $p = 2tm$ since m is sufficiently divisible. By the definition of volume, we have

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(-mK_X)) - (2t^2m^2 + tm)h^0(F, \mathcal{O}_F(-mK_F))}{m^3} \\ &= \frac{1}{6} \text{Vol}(-K_X) - 2t^2 \text{Vol}(-K_F) > 0. \end{aligned}$$

Note that $F \simeq \mathbb{P}^1$ and $\text{Vol}(-K_F) = 2$. Hence $h^0(X, \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^{2tm}) > 0$ for m sufficiently divisible. In particular, there exists an effective \mathbb{Q} -divisor $\Delta_F \sim_{\mathbb{Q}} -\frac{1}{t}K_X$ such that $\text{mult}_F \Delta_F \geq 2$. \square

A prime divisor V on X is *vertical* if $f(V)$ is a curve or V does not dominate S . Note that for a curve C on S passing through a general point, there is only one vertical prime divisor contained in $f^{-1}(C)$ and we denote it by V_C . It is easy to see that $f^*C = V_C$ as Weil divisors is well defined (since removing finitely many points of S , f is flat). For a general point $P \in S$, denote F_P be the fiber at P .

We can modify the \mathbb{Q} -divisor Δ_F to control vertical divisors by the following lemma.

Lemma 6.5. *For a general point $P \in S$, there exist an effective \mathbb{Q} -divisor $B_P \sim_{\mathbb{Q}} -\frac{a_P}{t}K_X$ for some $a_P \leq 384N(\epsilon)/\epsilon$ such that*

- (i) $\text{mult}_{F_P} B_P \geq 2$, and
- (ii) *For any curve C passing through P , $\text{mult}_{V_C} B_P \leq \epsilon/2$.*

Proof. Write $\Delta_{F_P} = \Delta_0 + \sum_i b_i V_{C_i}$ where Δ_0 does not contain vertical divisors passing through F_P and C_i is a curve passing through P . If $b_i \leq \epsilon/2$ for all i , then we can take $B_P = \Delta_{F_P}$ with $a_P = 1$.

Now we assume that $b_1 > \epsilon/2$. Note that by assumption, $(C_1)^2 > 0$ and $N(\epsilon)C_1$ is Cartier. So $N(\epsilon)C_1$ is a nef and big Cartier divisor. Hence by Kollár's Effective Base Point Free Theorem (see [18, 1.1 Theorem]), $|96N(\epsilon)C_1|$ is base point free. It follows that $|96N(\epsilon)C_1|$ defines a generically finite map $\Phi : S \rightarrow \mathbb{P}(|96N(\epsilon)C_1|)$. Since $P \in C_1$, P is not on the contracted curves of Φ . Hence by taking the pull back of general hyperplanes passing through $\Phi(P)$, we can write $96N(\epsilon)C_1 \sim_{\mathbb{Q}} \sum_j h_j H_j$ with $96N(\epsilon)C_1 \sim H_j$ a general curve passing through P , $0 < h_j \leq \epsilon/4$ for all j , and $\sum_j h_j = 1$. For $i > 1$, since C_i is semi-ample, we may take $C_i \sim_{\mathbb{Q}} D_i$ such that D_i is an effective \mathbb{Q} -divisor on S passing through a general point but not containing P . Now define

$$B_P := \frac{192N(\epsilon)}{b_1} \left(\Delta_0 + \sum_{i>1} b_i f^* D_i \right) + 2 \sum_j h_j f^* H_j \sim_{\mathbb{Q}} -\frac{192N(\epsilon)}{b_1 t} K_X,$$

and we can take $a_P = 192N(\epsilon)/b_1 \leq 384N(\epsilon)/\epsilon$. Note that

$$\text{mult}_{F_P} B_P \geq \text{mult}_P \left(2 \sum_j h_j H_j \right) \geq 2.$$

And by construction, for every curve C passing through P , if $C = H_j$ for some j , then $\text{mult}_{V_C} B_P = 2h_j \leq \epsilon/2$; otherwise $\text{mult}_{V_C} B_P = 0$.

Hence we proved the lemma. \square

Take two general points P_1 and P_2 on S . For simplicity, for $i = 1, 2$, we denote $F_{P_i} = F_i$, $a_{P_i} = a_i$, and $B_{P_i} = B_i$. Note that by construction,

$$F_i \subset \text{Nklt}(X, B_i).$$

Case 1. There exists a non-klt center E of dimension 2 of (X, B_i) containing F_i for some $i = 1$ or 2 .

In this case,

$$\text{mult}_E(B_i) \geq 1.$$

By construction of B_i , E is not vertical. Restricting on a general fiber F of f , we have

$$\frac{2a_i}{t} = -\frac{a_i}{t} K_X \cdot F = B_i \cdot F \geq E \cdot F \geq 1.$$

Hence

$$t \leq 2a_i \leq \frac{768N(\epsilon)}{\epsilon}.$$

Case 2. F_i is a maximal non-klt center of (X, B_i) for $i = 1$ and 2 .

Since P_1 is a general point, we may assume $F_1 \not\subset \text{Supp}(\Delta + B_2)$. Hence F_1 is a maximal non-klt center of $(X, (1 - \frac{a_1+a_2}{t})\Delta + B_1 + B_2)$ and F_2 is a non-klt center. Note that

$$-(K_X + (1 - \frac{a_1+a_2}{t})\Delta + B_1 + B_2) \sim_{\mathbb{Q}} -(1 - \frac{a_1+a_2}{t})(K_X + \Delta)$$

is ample by the assumption $t > 768N(\epsilon)/\epsilon$. By Connectedness Lemma, $\text{Nklt}(X, (1 - \frac{a_1+a_2}{t})\Delta + B_1 + B_2)$ is connected. Hence there is a non-klt center W intersecting with F_1 . Hence we have

$$\text{mult}_W \left(\left(1 - \frac{a_1+a_2}{t} \right) \Delta + B_1 + B_2 \right) \geq 1.$$

If $\dim W = 2$, since $(X, (1 - \frac{a_1+a_2}{t})\Delta)$ is ϵ -klt,

$$\text{mult}_W \left(\left(1 - \frac{a_1+a_2}{t} \right) \Delta \right) < 1 - \epsilon.$$

Hence

$$\text{mult}_W(B_1 + B_2) \geq \epsilon.$$

Since $F_1 \not\subset W$ by the maximality of F_1 , W is not vertical. Restricting on a general fiber F of f , we have

$$\frac{2(a_1+a_2)}{t} = -\frac{a_1+a_2}{t} K_X \cdot F = (B_1 + B_2) \cdot F \geq \epsilon W \cdot F \geq \epsilon.$$

Hence

$$t \leq \frac{2(a_1+a_2)}{\epsilon} \leq \frac{1536N(\epsilon)}{\epsilon^2}.$$

If $\dim W = 1$, then since P_1 is general, we may assume that for each point $Q \in F_1 \cap \text{Supp}(\Delta + B_2)$, Q is not contained in the singular locus of $\text{Supp}(\Delta + B_2)$. This is because the singular locus of $\text{Supp}(\Delta + B_2)$ has

dimension at most 1 and hence does not dominate S . Now if $W \subset \text{Supp}\Delta$, then W is contained in exactly one component of Δ since $\text{Supp}\Delta$ is smooth at points in $F_1 \cap W$. Since (X, Δ) is ϵ -klt, the coefficients of Δ is smaller than $1 - \epsilon$. Hence

$$\text{mult}_W \Delta < 1 - \epsilon.$$

Of course this inequality also holds if $W \not\subset \text{Supp}\Delta$. So we have

$$\text{mult}_W(B_1 + B_2) \geq \epsilon.$$

Note that to compute the intersection number $(B_1 + B_2) \cdot F$ for some fiber F by $\text{mult}_W(B_1 + B_2)$, it is necessary to avoid $V_{f(W)}$ in $B_1 + B_2$. Denote $V_{f(W)}$ by V . By construction of B_1 , $\text{mult}_V B_1 \leq \epsilon/2$. On the other hand, $\text{mult}_V B_2 = 0$ since $F_1 \not\subset \text{Supp}B_2$ but $F_1 \subset V$. We can write $B_1 + B_2 = B + \lambda V$ where the support of B does not contain V . Then $\lambda \leq \epsilon/2$. It is easy to see that $\text{mult}_W V = 1$ and hence

$$\text{mult}_W B = \text{mult}_W(B_1 + B_2) - \lambda \geq \frac{\epsilon}{2}.$$

Restricting on a fiber F of f at a general point of $f(W)$, we have

$$\frac{2(a_1 + a_2)}{t} = -\frac{a_1 + a_2}{t} K_X \cdot F = (B_1 + B_2) \cdot F = B \cdot F \geq \frac{\epsilon}{2}.$$

Hence

$$t \leq \frac{4(a_1 + a_2)}{\epsilon} \leq \frac{3072N(\epsilon)}{\epsilon^2}.$$

In summary, we have

$$t \leq \frac{4(a_1 + a_2)}{\epsilon} \leq \frac{3072N(\epsilon)}{\epsilon^2},$$

and hence

$$\text{Vol}(-K_X) \leq \frac{24 \cdot 3072^2 N(\epsilon)^2}{\epsilon^4}.$$

We have completed the proof of Theorem 2.4.

7. BOUNDEDNESS OF LOG FANO THREEFOLDS OF FIXED INDEX

In this section, we prove the boundedness of log Fano threefolds of fixed index (Corollary 1.8). Corollary 1.8 follows directly by Theorem 1.7 and the following more general theorem which might be well known to experts.

Theorem 7.1. *Fix positive integers r and n . Assume Weak BAB Conjecture holds in dimension n .*

Let \mathcal{D} be the set of all normal projective varieties X , where $\dim X = n$, K_X is \mathbb{Q} -Cartier, and there exists an effective \mathbb{Q} -divisor Δ such that (X, Δ) is klt and $-r(K_X + \Delta)$ is Cartier and ample.

Then \mathcal{D} forms a bounded family.

Proof. Consider a klt log Fano pair (X, Δ) of dimension n such that K_X is \mathbb{Q} -Cartier and $-r(K_X + \Delta)$ is Cartier.

Note that (X, Δ) is ϵ -klt with $\epsilon = 1/2r$ by the assumption. It follows that $-(K_X + \Delta)^n \leq M(n, \epsilon)$ by Weak BAB Conjecture in dimension n .

Since $-r(K_X + \Delta)$ is Cartier and ample, by Kollár's Effective Base Point Free Theorem [18, 1.1 Theorem, 1.2 Lemma], $G := -Nr(K_X + \Delta)$ is very ample for $N = 2n \cdot (n + 3)!$.

Note that $(K_X + G) \cdot C \geq 0$ for all curves C satisfying $K_X \cdot C \geq -2n$. Hence by Cone Theorem (see [23, Theorem 3.7]), $K_X + G$ is nef.

Now we can bound G^n and $|-K_X \cdot G^{n-1}|$ from above. Clearly $G^n \leq N^n r^n M(n, \epsilon)$ by definition and $-K_X \cdot G^{n-1} > 0$ since $-K_X$ is big. On the other hand,

$$\begin{aligned} & -K_X \cdot G^{n-1} \\ &= -(K_X + G) \cdot G^{n-1} + G \cdot G^{n-1} \\ &\leq N^n r^n M(n, \epsilon). \end{aligned}$$

Hence G^n and $|-K_X \cdot G^{n-1}|$ are bounded from above. By [20], the coefficients of the Hilbert polynomial $P(t) = \chi(X, \mathcal{O}_X(tG))$ is bounded and hence there are only finitely many Hilbert polynomials for the polarized variety (X, G) . And hence X is in a bounded family.

We complete the proof. \square

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